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ON THE ABSENCE OF ESHELBY PROPERTY FOR NON-ELLIPSOIDAL INCLUSIONS

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Abstract—It is shown that the Eshelby property does not hold for any inclusion bounded by a polynomial surface of higher than the second-degree, or any inclusion bounded by a non-convex surface. Inclusions bounded by segments of two or more different surfaces are also precluded. The absence of the Eshelby property for non-ellipsoidal inclusions is then discussed.

1. INTRODUCTION

Eshelby (1957, 1961) has shown that the strain field within a homogeneous ellipsoidal inclusion in an infinite elastic matrix is uniform, if the eigenstrain in the inclusion is uniform. He also stated that among finite inclusions the ellipsoidal alone has this convenient (Eshelby) property. Attempts were made to find other shapes of inclusions with this property (Mura *et al.*, 1994), and Rodin (1996) showed by an elaborate calculation that polyhedral inclusions cannot have such a property. Markenscoff (1997a) provided an alternative proof for this, and showed that the external field is fully determined by any small analytic part of the boundary, since there, the constancy of stress in the interior domain renders the exterior problem overdetermined. She also showed (Markenscoff, 1997b) that the only perturbations of the ellipsoid that maintain the Eshelby property are those into another ellipsoid. In this paper we show that the Eshelby property does not hold for any inclusion bounded by a polynomial surface of higher than the second-degree, or any inclusion bounded by segments of two or more different surfaces. Simple examples for the latter are inclusions bounded by a plane and an ellipsoidal surface, or any polyhedral inclusion. The proof that an inclusion bounded by a non-convex surface cannot have the Eshelby property is also given. The absence of the Eshelby property for non-ellipsoidal inclusions is then discussed.

2. ESHELBY PROPERTY

If an inclusion removed from an infinite elastic matrix undergoes a stress free uniform eigenstrain ε_{ij}^*, the surface traction required to bring it back to its original size and shape, when outside the matrix, is $-\sigma_{ij}^* n_j$. The outer normal to the boundary of the inclusion has the components n_i, and $\sigma_{ij}^* = C_{ijkl}\varepsilon_{kl}^*$. The elastic constants of a homogeneous material are C_{ijkl}. Upon insertion back into the matrix, the stress in the inclusion is partially relaxed, which is accompanied by an associated build-up of stress in the surrounding matrix. The inclusion will have the Eshelby property of constant state of stress and strain if its bounding surface S is such that the following holds:

A layer of body forces $\sigma_{ij}^* n_j\,\mathrm{d}S$, where σ_{ij}^* is a uniform stress state, distributed over the surface S in an infinite medium, produces a linear displacement distribution of the points of S, given by $u_i(\mathbf{x}) = a_{ij}x_j + b_i$.

By the uniqueness theorem of elasticity, the displacement distribution in the material inside the surface S is also linear, so that the strain and rotation within S are both uniform

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and given by symmetric and antisymmetric parts of a_{ij}, respectively. The Eshelby property is an immediate consequence.

3. DISPLACEMENT EXPRESSION

The displacement at an arbitrary point $\mathbf{x}$ on the surface S due to a layer of body forces $\sigma^*_{ij}n_j\,\mathrm{d}S$ distributed over S is obtained by integration from the Green's function as

$$u_i(\mathbf{x}) = \frac{\sigma^*_{jk}}{16\pi\mu(1-\nu)}\int_S \frac{1}{r}[(3-4\nu)\delta_{ij}+l_il_j]n_k\,\mathrm{d}S, \tag{1}$$

where $r = |\mathbf{X}-\mathbf{x}|$, and $l_i = (X_i - x_i)/r$ (Fig. 1). The Kronecker delta is denoted by δ_{ij}, and μ and ν are the shear modulus and Poisson's ratio of the isotropic elastic material. When S is a closed convex surface, eqn (1) can be conveniently rewritten as (Eshelby, 1957)

$$u_i(\mathbf{x}) = -\frac{\sigma^*_{jk}}{16\pi\mu(1-\nu)}\int_{2\pi} f_{ijk}r\,\mathrm{d}\Omega. \tag{2}$$

The angle Ω is a solid angle, and

$$f_{ijk} = (1-2\nu)(\delta_{ik}l_j+\delta_{ij}l_k)-\delta_{jk}l_i+3l_il_jl_k. \tag{3}$$

The integration in eqn (2) extends over only half of the total solid angle of 4π, since the entire convex surface S can be seen from any point $\mathbf{x}$ on S within a solid angle of 2π. For a non-convex surface, the angle through which it can be seen from a point on S can be different from 2π, and can depend on $\mathbf{x}$. This is discussed in more detail later in the paper. If rf_{ijk} in eqn (2) is an even function of l_i, the integral over the solid angle of 2π can be

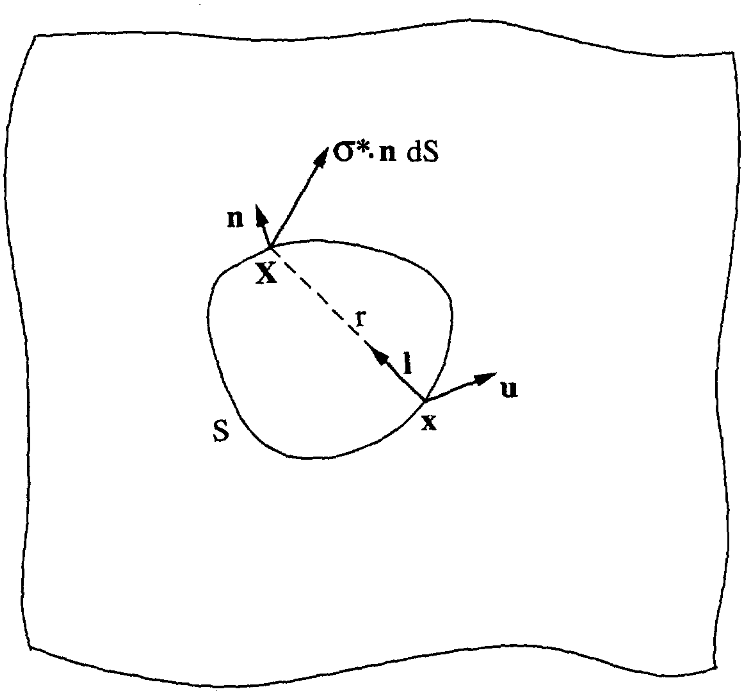

Fig. 1. An internal surface S within an infinite medium caring a layer of distributed body forces $\sigma^*\cdot\mathbf{n}\,\mathrm{d}S$. Displacement at an arbitrary point $\mathbf{x}$ on S is $\mathbf{u}$. The distance between the points $\mathbf{x}$ and $\mathbf{X}$ is r, and its direction $\mathbf{l}$.

calculated as one half of the integral over the solid angle of 4π. Thus, for given l_i, the displacements u_i in eqn (2) will be linear function of x_i if r is linear in x_i.

4. THE SHAPE OF AN INCLUSION

If a bounding surface of the inclusion is an m-degree surface $F(\mathbf{x}) = 0$, then for any other point $\mathbf{X} = \mathbf{x}+r\mathbf{l}$ on S,

$$F(\mathbf{x}+r\mathbf{l}) = F(\mathbf{x})+\frac{\partial F}{\partial x_i} r l_i + \frac{1}{2!}\frac{\partial^2 F}{\partial x_i \partial x_j} r^2 l_i l_j + \cdots + \frac{1}{m!}\frac{\partial^m F}{\partial x_i \partial x_j \ldots \partial x_m} r^m l_i l_j \ldots l_m. \tag{4}$$

Since $F(\mathbf{x}) = F(\mathbf{X}) = 0$, eqn (4) gives

$$\frac{\partial F}{\partial x_i} l_i + \frac{1}{2!}\frac{\partial^2 F}{\partial x_i \partial x_j} r l_i l_j + \cdots + \frac{1}{m!}\frac{\partial^m F}{\partial x_i \partial x_j \ldots \partial x_m} r^{m-1} l_i l_j \ldots l_m = 0. \tag{5}$$

Closed surfaces correspond to even m, i.e., even degree of the highest power in the polynomial representation of F. Clearly, for given l_i, r is a linear function of x_i if S is a quadratic surface

$$F(\mathbf{x}) = A_{ij}x_i x_j + B_i x_i - C = 0, \tag{6}$$

for then

$$r = -\frac{(2A_{ij}x_j + B_i)l_i}{A_{\alpha\beta}l_\alpha l_\beta}. \tag{7}$$

It is assumed that the Hessian matrix A_{ij} is positive definite, so that eqn (6) corresponds to an ellipsoidal surface. Thus, r in eqn (7) is positive, since the unit vector with components l_i is directed from $\mathbf{x}$ inwards the convex surface S, while $2A_{ij}x_j + B_i = \partial F/\partial x_i$ is normal to S and directed outside of it. Substitution of eqn (7) into eqn (2), therefore, gives

$$u_i(\mathbf{x}) = a_{ij}x_j + b_i. \tag{8}$$

The constants

$$a_{ij} = \frac{\sigma^*_{sk}}{8\pi\mu(1-\nu)}\int_{2\pi} f_{isk}\frac{A_{j\gamma}l_\gamma}{A_{\alpha\beta}l_\alpha l_\beta}\,\mathrm{d}\Omega, \tag{9}$$

$$b_i = \frac{\sigma^*_{sk}}{16\pi\mu(1-\nu)}\int_{2\pi} f_{isk}\frac{B_\gamma l_\gamma}{A_{\alpha\beta}l_\alpha l_\beta}\,\mathrm{d}\Omega \tag{10}$$

depend only on the eigenstrain, Poisson's ratio, and the shape of the ellipsoid. The symmetric part of a_{ij} is the uniform strain within S, the antisymmetric part is the uniform rotation, and b_i is a possible rigid-body translation. The above integrals can be evaluated explicitly, and the strain within S can be expressed as $\varepsilon^0_{ij} = \mathcal{S}_{ijkl}\varepsilon^*_{kl}$, where $\mathcal{S}_{ijkl}$ are the components of the (uniform) Eshelby tensor. This provides an alternative proof to Eshelby's result for ellipsoidal inclusion (Eshelby 1957, 1961). It should be mentioned that for a point $\mathbf{x}$ on the surface of the ellipsoid, the distance to any other point on the ellipsoid is $r = -2f/g$, where, in Eshelby's (1957) notation, $f = A_{ij}l_i x_j$, $g = A_{ij}l_i l_j$ [and $B_i = 0$ in our eqn (7)]. Therefore, while Eshelby's eqn (3.2) applies to $\mathbf{x}$ inside the ellipsoid, it does not apply to $\mathbf{x}$ on the ellipsoidal surface, since for $e = 0$ it gives $r = 0$. Since the required integration of rf_{ijk} for $\mathbf{x}$ on the surface of the ellipsoid spans only half of the total solid angle, the same values for the Eshelby tensor components are obtained from our eqn (9) as from Eshelby's eqn (3.4), derived for an interior $\mathbf{x}$ and in which the integration spans the total solid angle of 4π.

There is no third-degree surface that is closed by itself, but a part of the bounding surface of a finite inclusion can be a third-degree surface. It is thus, instructive to consider the cubic representation of F, because it is the simplest one to demonstrate that the Eshelby property cannot hold for an inclusion bounded by a polynomial surface of higher than the second-degree. Thus, for $m = 3$,

$$F(\mathbf{x}) = A_{ijk}x_ix_jx_k + B_{ij}x_ix_j + C_ix_i - D = 0, \tag{11}$$

and from eqn (5)

$$r = -\frac{b}{2a} + \left[\left(\frac{b}{2a}\right)^2 - \frac{c}{a}\right]^{1/2}, \tag{12}$$

where:

$$a = A_{ijk}l_il_jl_k, \tag{13}$$

$$b = (3A_{ijk}x_k + B_{ij})l_il_j, \tag{14}$$

$$c = (3A_{ijk}x_jx_k + 2B_{ij}x_j + C_i)l_i. \tag{15}$$

For a convex portion of the surface $F = 0$, c is negative, and for the positive definite tensors A_{ijk} and B_{ij}, both a and b are positive. Thus, the distance r in eqn (12) is also positive. However, while the first term $(-b/2a)$ is linear in x_i, the square-root term is a nonlinear function of the coordinates x_i. This implies that the contribution to displacements at the point $\mathbf{x}$ on $F = 0$, due to tractions applied over $F = 0$, cannot be linear in x_i. Consequently, the Eshelby property cannot hold for an inclusion bounded in part by a third-degree surface. Furthermore, the angle through which this surface is seen from a point $\mathbf{x}$ depends on $\mathbf{x}$, which also makes the displacements nonlinear functions of the coordinates x_i. It should be pointed out that the square-root term in the expression (12) for r is an even function of l_i, while f_{ijk} in eqn (3) are odd functions of l_i, but the integral of their contribution to the product of rf_{ijk} does not vanish, because the required integration does not extend over the total solid angle of 4π.

In the case of the fourth or higher-degree surface, r is also a nonlinear function of the coordinates x_i, and the Eshelby property does not hold for inclusions bounded by such surfaces, either. The non-convex surfaces are additionally precluded by reasons given in the next section.

5. OTHER SHAPES OF INCLUSIONS

The Eshelby property does not hold for an inclusion with a bounding surface that consists of the segments of two or more different surfaces. For example, consider an inclusion shown in Fig. 2 obtained by bisecting an ellipsoidal inclusion with a planar cut. For a point $\mathbf{x}$ on the ellipsoidal portion of the bounding surface ($F_1 = 0$), the displacement is

$$u_i(\mathbf{x}) = -\frac{\sigma^*_{jk}}{16\pi\mu(1-\nu)}\left(\int_{\Omega_1(\mathbf{x})} f_{ijk}r_1\,\mathrm{d}\Omega + \int_{\Omega_2(\mathbf{x})} f_{ijk}r_2\,\mathrm{d}\Omega + \int_{\Omega_3(\mathbf{x})} f_{ijk}r_3\,\mathrm{d}\Omega\right), \tag{16}$$

where the solid angles Ω_1, Ω_2 and Ω_3 depend on the location of the point $\mathbf{x}$. Since each of the r distances (r_1, r_2, r_3), from the point $\mathbf{x}$ on the surface $F_1 = 0$ to another point on that surface, or to the point on the plane $F_2 = 0$, is a linear function of x_i, eqn (16) gives upon

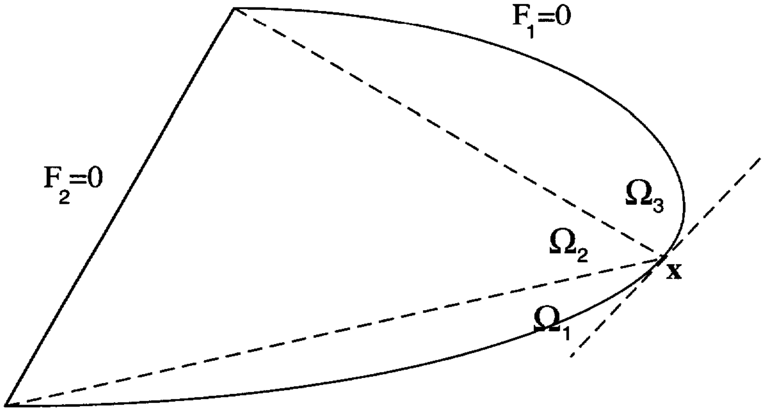

Fig. 2. An inclusion obtained from an ellipsoidal inclusion ($F_1 = 0$) by a planar cut ($F_2 = 0$). From a point **x** on the ellipsoidal part of the inclusion, the planar part is seen through an angle Ω_2, while the ellipsoidal parts are seen through the angles Ω_1 and Ω_3. Evidently, $\Omega_2 = \Omega_2(\mathbf{x})$, and likewise Ω_1 and Ω_3.

integration a nonlinear displacement distribution. When **x** is on the planar part of S, the distance r to the point on the ellipsoidal part of S is itself a nonlinear function of the coordinates x_i, and from eqn (2) it follows that the corresponding displacement is also a nonlinear function of the coordinates x_i. Therefore, the Eshelby property does not hold for the considered inclusion. This was expected on physical grounds, because the resistance of the surrounding matrix near the encounter of two surfaces is quite different than away from them, and the planar boundary would take a curved shape upon insertion of the inclusion, which in turn implies a nonuniform final state of strain within the inclusion. Similarly, the Eshelby property does not hold for an inclusion made by two halves of two different ellipsoids, which share the two equal semi-axes but have the third semi-axis different. Also, obviously, precluded are finite inclusions that are parts of elliptic cylinders, hyperboloids or paraboloids.

By extension of the analysis, it is straightforward to show that none of the polyhedral inclusions have the Eshelby property. Although the distance r from a point **x** on one side of a polyhedron to a point **X** on any other side of a polyhedron is a linear function of **x**, the solid angle through which each side of the polyhedron is seen from the point **x** depends on **x** (Fig. 3), and thus the displacements must be nonlinear functions of **x**. Therefore, for a uniform eigenstrain, the final state of strain within a polyhedral inclusion cannot be uniform. Indeed, a nonuniform strain in a cuboidal inclusion, associated with a uniform

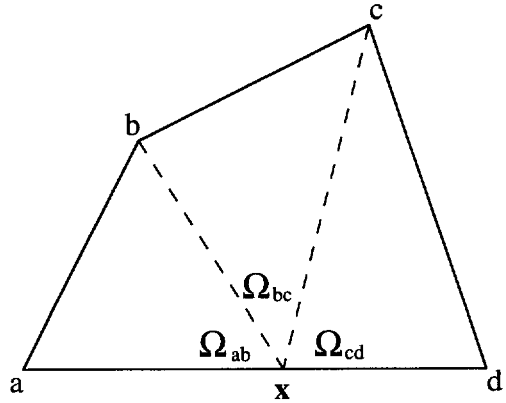

Fig. 3. The angle through which the side *ab* is seen from a point **x** on the side *ad* of a polyhedron *abcd* depends on **x**, $\Omega_{ab} = \Omega_{ab}(\mathbf{x})$. Likewise, $\Omega_{bc} = \Omega_{bc}(\mathbf{x})$ and $\Omega_{cd} = \Omega_{cd}(\mathbf{x})$.

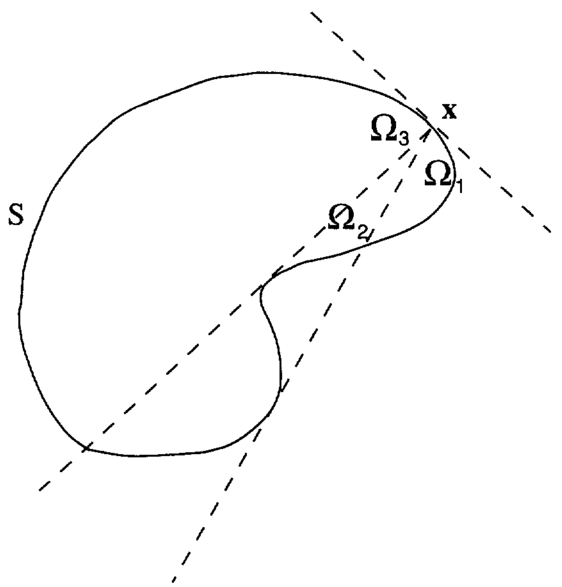

Fig. 4. The solid angles Ω_1 and Ω_3 define the parts of a non-convex bounding surface S for which the distance r from $\mathbf{x}$ is single-valued. Within the solid angle Ω_2, r is not single-valued and an adequate integration is needed to encompass the corresponding parts of the boundary S. Clearly, $\Omega_2 = \Omega_2(\mathbf{x})$, and likewise Ω_1 and Ω_3.

eigenstrain, was calculated by Faivre (1964). The reference to other related work can be found in Mura (1987).

By similar reasons, all inclusions with non-convex boundaries are precluded from having the Eshelby property. This is illustrated in Fig. 4. The solid angles Ω_1 and Ω_3 define the portions of S for which the distance r from $\mathbf{x}$ to S is single-valued. Since all three angles (Ω_1, Ω_2 and Ω_3) depend on $\mathbf{x}$, upon an appropriate integration it follows that displacement components are nonlinear functions of the coordinates x_i. Also, from some points of S the non-convex surface is seen within the solid angle of 2π, but from other points it is seen within a solid angle greater than 2π.

6. DISCUSSION

Among all considered shapes of inclusions, only the ellipsoidal inclusion has the Eshelby property of constant strain, as conjectured by Eshelby (1961). Referring to the auxiliary problem in Fig. 1, if the strain within S is uniform and equal to ε_{ij}^0, the associated part of the traction on S is $C_{ijkl}\varepsilon_{kl}^0 n_j$. The rest of the applied traction, $(\sigma_{ij}^* - C_{ijkl}\varepsilon_{kl}^0)n_j$, is transmitted to the surrounding matrix and strains the material outside of S. This part of the traction is, therefore, also deduced from a uniform state of stress, i.e., from the stress $\hat{\sigma}_{ij} = \sigma_{ij}^* - C_{ijkl}\varepsilon_{kl}^0$. Only an ellipsoid and an ellipsoidal cavity in an infinite medium, loaded by tractions that can be deduced from uniform states of stress, both deform into ellipsoidal shapes, and this leads to the Eshelby property for the ellipsoidal inclusion. Of course, the stress state at the points just outside the inclusion is nonuniform, but it is only a uniform part of it ($\hat{\sigma}_{ij}$) that contributes in the Cauchy relation to the traction transmitted to S and to the surrounding matrix. The stress $\hat{\sigma}_{ij}$ is not the average stress over S for the points just outside of S, which is actually equal to zero for a spherical inclusion. This can be deduced by an appropriate integration from eqns (2.13) of Eshelby (1957), or from the Tanaka and Mori (1972) observation that the average stress and strain in the region between the ellipsoidal inclusion and any ellipsoidal surface, concentric and similar in shape to the inclusion, are equal to zero. Therefore, the integral of the symmetric part of $u_i n_j$ over any such surface is constant and equal to $\varepsilon_{ij}^0 V$, where V is the volume of the inclusion.

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